Mocninné řady

Rozviňte funkci v mocninnou řadu a určete interval, na kterém rovnost platí

$$\mathrm{f}(x) = \arcsin 3x^2$$

Řešení

$$\mathrm{f}(x) = \arcsin 3x^2$$

$$\begin{aligned}
\mathrm{f}'(x) &= \frac{6x}{\sqrt{1-9x^4}} = 6x\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-9x^4)^n = 6x\sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}(-9)^n x^{4n} = \\
&= 6\sum_{n=0}^{\infty}9^n\frac{(2n-1)!!}{(2n)!!}x^{4n+1}
\end{aligned}$$

Využili jsme přitom toho, že

$$\begin{aligned}
\binom{-\frac{1}{2}}{n} &= \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\ldots\left(-\frac{1}{2}-n+1\right)}{n!} = (-1)^n\frac{\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)\ldots\left(\frac{2n-1}{2}\right)}{n!} = \\
&= \left(-\frac{1}{2}\right)^n\frac{1\cdot 3\ldots(2n-1)}{n!} = (-1)^n\frac{(2n-1)!!}{(2n)!!}
\end{aligned}$$

Derivaci mocninné řady zintegrujeme

$$\begin{aligned}
\mathrm{f}'(x) &= 6\sum_{n=0}^{\infty}9^n\frac{(2n-1)!!}{(2n)!!}x^{4n+1} \\
\mathrm{f}(x) &= 6\sum_{n=0}^{\infty}9^n\frac{(2n-1)!!}{(2n)!!}\frac{x^{4n+2}}{4n+2} + \mathrm{f}(0) = \\
&= 6\sum_{n=0}^{\infty}\frac{9^n}{4n+2}\frac{(2n-1)!!}{(2n)!!}x^{4n+2} + \arcsin 0 = \\
&= 6\sum_{n=0}^{\infty}\frac{9^n}{4n+2}\frac{(2n-1)!!}{(2n)!!}x^{4n+2}
\end{aligned}$$

Vypočítáme poloměr konvergence

$$a_n = \frac{(2n-1)!!\ 9^n}{(4n+2)(2n)!!}$$

$$\begin{aligned}
r_1 = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| &= \lim_{n\to\infty}\left|\frac{\frac{(2n-1)!!\ 9^n}{(4n+2)(2n)!!}}{\frac{(2n+1)!!\ 9^{n+1}}{(4n+6)(2n+2)!!}}\right| = \\
&= \lim_{n\to\infty}\left|\frac{(2n-1)!!\ 9^n}{(4n+2)(2n)!!}\frac{(4n+6)(2n+2)(2n)!!}{(2n+1)(2n-1)!!\ 9^{n+1}}\right| = \\
&= \lim_{n\to\infty}\left|\frac{1}{9}\frac{(4n+6)(2n+2)}{(4n+2)(2n+1)}\right| = \\
&= \frac{1}{9}\lim_{n\to\infty}\left|\frac{n^2\left(4+\frac{6}{n}\right)\left(2+\frac{2}{n}\right)}{n^2\left(4+\frac{2}{n}\right)\left(2+\frac{1}{n}\right)}\right| = \frac{1}{9}
\end{aligned}$$

$$\left|\frac{x^{4n+6}}{x^{4n+2}}\right| = \left|x^4\right| < r_1 = \frac{1}{9}$$

$$\left|x^4\right| < \frac{1}{9}$$

$$|x| < \sqrt{\frac{1}{3}} = r$$

Víme tedy, že pro $x \in \left(-\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}}\right)$ řada absolutně konverguje a rovnost

$$\mathrm{f}(x) = \arcsin 3x^2 = 6\sum_{n=0}^{\infty} \frac{9^n}{4n+2} \frac{(2n-1)!!}{(2n)!!} x^{4n+2}$$

zde platí.

Vyšetříme ještě krajní body intervalu
$x = -\sqrt{\frac{1}{3}}$:

$$\begin{aligned}
6\sum_{n=0}^{\infty} \frac{9^n}{4n+2} \frac{(2n-1)!!}{(2n)!!} x^{4n+2} &= 6\sum_{n=0}^{\infty} \frac{9^n}{4n+2} \frac{(2n-1)!!}{(2n)!!} \left(-\sqrt{\frac{1}{3}}\right)^{4n+2} = \\
&= 6\sum_{n=0}^{\infty} \frac{9^n}{4n+2} \frac{(2n-1)!!}{(2n)!!} \left(\sqrt{\frac{1}{3}}\right)^{4n} \frac{1}{3} = \\
&= 2\sum_{n=0}^{\infty} \frac{9^n}{4n+2} \frac{(2n-1)!!}{(2n)!!} \frac{1}{9^n} = \\
&= 2\sum_{n=0}^{\infty} \frac{(2n-1)!!}{(4n+2)(2n)!!}
\end{aligned}$$

Podle podílového kritéria nelze rozhodnout, aplikujme tedy Raabeovo kritérium

$$\begin{aligned}
\lim_{n\to\infty} n\left(1 - \frac{a_{n+1}}{a_n}\right) &= \lim_{n\to\infty} n\left(1 - \frac{\frac{(2n+1)!!}{(2n+2)!!(4n+6)}}{\frac{(2n-1)!!}{(2n)!!(4n+2)}}\right) = \\
&= \lim_{n\to\infty} n\left(1 - \frac{(2n+1)(2n-1)!!}{(2n+2)(2n)!!(4n+6)} \frac{(2n)!!(4n+2)}{(2n-1)!!}\right) = \\
&= \lim_{n\to\infty} n\left(1 - \frac{(2n+1)(4n+2)}{(2n+2)(4n+6)}\right) = \\
&= \lim_{n\to\infty} n\left(1 - \frac{8n^2+8n+2}{8n^2+20n+12}\right) = \\
&= \lim_{n\to\infty} n\left(\frac{12n+10}{8n^2+20n+12}\right) = \\
&= \lim_{n\to\infty} \frac{12n^2+10n}{8n^2+20n+12} = \\
&= \lim_{n\to\infty} \frac{n^2\left(12+\frac{10}{n}\right)}{n^2\left(8+\frac{20}{n}+\frac{12}{n^2}\right)} = \\
&= \lim_{n\to\infty} \frac{12}{8} = \frac{3}{2} > 1
\end{aligned}$$

Z Raabeova kritéria vyplývá, že řada pro bod $x = -\sqrt{\frac{1}{3}}$ konverguje a rovnost zde platí. Dále zjistíme, zda konverguje i pro $x = \sqrt{\frac{1}{3}}$:

$$6\sum_{n=0}^{\infty}\frac{9^n}{4n+2}\frac{(2n-1)!!}{(2n)!!}x^{4n+2} = 6\sum_{n=0}^{\infty}\frac{9^n}{4n+2}\frac{(2n-1)!!}{(2n)!!}\left(\sqrt{\frac{1}{3}}\right)^{4n+2} = 2\sum_{n=0}^{\infty}\frac{(2n-1)!!}{(4n+2)(2n)!!}$$

Řada je stejná jako pro $x = -\sqrt{\frac{1}{3}}$, bude tedy konvergovat a rovnost zde platí také.

Rovnost

$$\mathrm{f}(x) = \arcsin 3x^2 = 6\sum_{n=0}^{\infty}\frac{9^n}{(4n+2)}\frac{(2n-1)!!}{(2n)!!}x^{4n+1}$$

tedy platí na intervalu $\forall x \in \left\langle -\sqrt{\frac{1}{3}}, \sqrt{\frac{1}{3}} \right\rangle$.