Mocninné řady

Rozviňte funkci v mocninnou řadu a určete interval, na kterém rovnost platí

$$\mathrm{f}(x) = x \arcsin 3x + \frac{1}{3}\sqrt{1-9x^2}$$

Řešení

$$\mathrm{f}(x) = x \arcsin 3x + \frac{1}{3}\sqrt{1-9x^2}$$

$$\mathrm{f}'(x) = \arcsin 3x + \frac{3x}{\sqrt{1-9x^2}} - \frac{3x}{\sqrt{1-9x^2}} = \arcsin 3x$$

$$\begin{aligned}\mathrm{f}''(x) = \frac{3}{\sqrt{1-9x^2}} = 3(1-9x^2)^{-\frac{1}{2}} = 3\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-9x^2)^n = 3\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n 3^{2n}x^{2n} = \\ = \sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}(-1)^n 3^{2n+1}x^{2n} = \sum_{n=0}^{\infty}(-1)^n\frac{(2n-1)!!}{(2n)!!}(-1)^n 3^{2n+1}x^{2n} \\ = \sum_{n=0}^{\infty}(-1)^{2n}\frac{(2n-1)!!}{(2n)!!}3^{2n+1}x^{2n} = \sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}3^{2n+1}x^{2n}\end{aligned}$$

Využili jsme přitom toho, že

$$\begin{aligned}\binom{-\frac{1}{2}}{n} = \frac{(-\frac{1}{2})(-\frac{3}{2})\dots(-\frac{1}{2}-n+1)}{n!} = \frac{(-\frac{1}{2})(-\frac{3}{2})\dots(-n+\frac{1}{2})}{n!} = \frac{(-\frac{1}{2})^n 1\cdot 3\dots(2n-1)}{n!} = \\ = \frac{(-\frac{1}{2})^n(2n-1)!!}{n!} = \frac{(-1)^n(2n-1)!!}{2^n n!} = \frac{(-1)^n(2n-1)!!}{(2n)!!} = (-1)^n\frac{(2n-1)!!}{(2n)!!}\end{aligned}$$

Derivace mocninných řad zintegrujeme

$$\begin{aligned}\mathrm{f}''(x) &= \sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}3^{2n+1}x^{2n} \\ \mathrm{f}'(x) &= \sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}3^{2n+1}\frac{x^{2n+1}}{2n+1} + f'(0) \\ &= \sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}3^{2n+1}\frac{x^{2n+1}}{2n+1} + \arcsin(3\cdot 0) \\ &= \sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}n3^{2n+1}\frac{x^{2n+1}}{2n+1} + 0 \\ &= \sum_{n=0}^{\infty}\frac{(2n-1)!!}{(2n)!!}3^{2n+1}\frac{x^{2n+1}}{2n+1}\end{aligned}$$

$$
\begin{aligned}
\mathrm{f}(x) &= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} 3^{2n+1} \frac{x^{2n+2}}{2n+2} \frac{1}{2n+1} + f(0) \\
&= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} 3^{2n+1} \frac{x^{2n+2}}{2n+2} \frac{1}{2n+1} + 0 \cdot \arcsin(3 \cdot 0) + \frac{1}{3}\sqrt{1 - 9 \cdot 0} \\
&= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} 3^{2n+1} \frac{x^{2n+2}}{2n+2} \frac{1}{2n+1} + \frac{1}{3}
\end{aligned}
$$

Vypočítáme poloměr konvergence

$$
\begin{aligned}
r_1 &= \lim_{n\to\infty} \left| \frac{\frac{(2n-1)!!3^{2n+1}}{(2n+2)!!(2n+1)}}{\frac{(2n+1)!!3^{2n+3}}{(2n+4)!!(2n+3)}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(2n-1)!!3^{2n+1}(2n+4)!!(2n+3)}{(2n+2)!!(2n+1)(2n+1)!!3^{2n+3}} \right| \\
&= \lim_{n\to\infty} \left| \frac{(2n-1)!!3^{2n}3(2n+4)(2n+2)!!(2n+3)}{(2n+2)!!(2n+1)(2n-1)!!3^{2n}3^3} \right| \\
&= \lim_{n\to\infty} \left| \frac{(2n+4)(2n+3)}{(2n+1)(2n+1)9} \right| \\
&= \lim_{n\to\infty} \left| \frac{4n^2+14n+12}{36n^2+36n+9} \right| \\
&= \lim_{n\to\infty} \left| \frac{4n^2(1+\frac{14n}{4n^2}+\frac{12}{4n^2})}{4n^2(9+\frac{36n}{4n^2}+\frac{9}{4n^2})} \right| \\
&= \frac{1}{9}
\end{aligned}
$$

$$
\begin{aligned}
\left| \frac{x^{2n+4}}{x^{2n+2}} \right| = \left| x^2 \right| &< r_1 = \frac{1}{9} \\
|x| &< \sqrt{\frac{1}{9}} = \frac{1}{3} = r
\end{aligned}
$$

Víme tedy, že pro $x \in (-\frac{1}{3}, \frac{1}{3})$ řada absolutně konverguje a rovnost

$$
\mathrm{f}(x) = x \arcsin 3x + \frac{1}{3}\sqrt{1-9x^2} = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} 3^{2n+1} \frac{x^{2n+2}}{2n+2} \frac{1}{2n+1} + \frac{1}{3}
$$

zde platí.

Vyšetříme ještě krajní body intervalu

$x = \frac{1}{3}$

$$\sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n+2)!!} \frac{3^{2n+1}}{2n+1} \left(\frac{1}{3}\right)^{2n+2} + \frac{1}{3} = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n+2)!!} \frac{3^{2n}3}{2n+1} \left(\frac{1}{3}\right)^{2n} \frac{1}{3}\frac{1}{3} + \frac{1}{3}$$
$$= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{3(2n+2)!!(2n+1)} + \frac{1}{3}$$

Použijeme Gaussovo kritérium, které říká

$$\frac{a_{n+1}}{a_n} = \lambda - \frac{\mu}{n} + \frac{v_n}{n^\gamma}$$
$$\lambda, \mu, v_n, \gamma \in \mathbb{R}; \gamma > 1$$

také musí platit, že $\{v_n\}_{n=1}^{\infty}$ je omezená posloupnost.
Řada potom pro:
$\lambda < 1 \ldots$ konverguje
$\lambda > 1 \ldots$ diverguje
$\lambda = 1 \wedge \mu > 1 \ldots$ konverguje
$\lambda = 1 \wedge \mu \leq 1 \ldots$ diverguje

$$\frac{a_{n+1}}{a_n} = \frac{\frac{(2n+1)!!}{3(2n+4)!!(2n+3)}}{\frac{(2n-1)!!}{3(2n+2)!!(2n+1)}} = \frac{3(2n+1)!!(2n+2)!!(2n+1)}{3(2n+4)!!(2n+3)(2n-1)!!} = \frac{(2n+1)(2n-1)!!(2n+2)!!(2n+1)}{(2n+4)(2n+2)!!(2n+3)(2n-1)!!} =$$
$$= \frac{(2n+1)(2n+1)}{(2n+4)(2n+3)} = \frac{4n^2+4n+1}{4n^2+14n+12} = 1 - \frac{5}{2n} + \frac{24+\frac{30}{n}}{4n^2+14n+12} = 1 - \frac{5}{2n} + \frac{1}{n^2}\frac{24+\frac{30}{n}}{4+\frac{14}{n}+\frac{12}{n^2}}$$

Dostáváme:
$\lambda = 1$
$\mu > 1$
$v_n = \frac{24+\frac{30}{n}}{4+\frac{14}{n}+\frac{12}{n^2}}$

$$\lim_{n\to\infty} v_n = \lim_{n\to\infty} \frac{24+\frac{30}{n}}{4+\frac{14}{n}+\frac{12}{n^2}} = \frac{24}{4} = 6$$

$\{v_n\}_{n=1}^{\infty}$ je omezená

Podle Gaussova kritéria řada pro $x = \frac{1}{3}$ konverguje

Nyní vyšetšíme jak se řada chová pro $x = -\frac{1}{3}$

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n+2)!!} \frac{3^{2n+1}}{2n+1} \left(-\frac{1}{3}\right)^{2n+2} + \frac{1}{3} &= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n+2)!!(2n+1)} \frac{1}{3} + \frac{1}{3} \\ &= \sum_{n=0}^{\infty} \frac{(2n-1)!!}{3(2n+2)!!(2n+1)} + \frac{1}{3}\end{aligned}$$

Řada se chová stejně jako pro $x = \frac{1}{3}$

Proto platí, že pro $x \in \langle -\frac{1}{3}, \frac{1}{3} \rangle$ řada absolutně konverguje a rovnost

$$\mathrm{f}(x) = x \arcsin 3x + \frac{1}{3}\sqrt{1-9x^2} = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{(2n)!!} 3^{2n+1} \frac{x^{2n+2}}{2n+2} \frac{1}{2n+1} + \frac{1}{3}$$

zde platí.