Diferenciální rovnice 2. řádu

Speciální případ $\mathrm{F}(x, y', y'') = 0$

$$(x+2)y'' - y' = 2xy'^2$$

Řešení

Použijeme substituci pro převod na diferenciální rovnici 1.řádu

$$\begin{aligned} y' &= z \\ y'' &= z' \end{aligned}$$

$$\begin{aligned} (x+2)z' - z &= 2xz^2 \quad / : z^2, \quad z^2 \not\equiv 0 \\ & \qquad z \equiv 0 \Rightarrow z' \equiv 0 \\ & \qquad (x+2) \cdot 0 - 0 = 2x \cdot 0 \\ & \qquad 0 = 0 \\ & \qquad z \equiv 0 \text{ je řešení} \\ (x+2)z^{-2}z' - z^{-1} &= 2x \end{aligned}$$

Použijeme substituci

$$\begin{aligned} z^{-1} &= u \\ -1z^{-2}z' &= u' \\ z^{-2}z' &= -u' \end{aligned}$$

$$\begin{aligned} -(x+2)u' - u &= 2x \quad / : (-(x+2)), \quad x \neq -2 \\ & \qquad x = -2 \\ & \qquad -0 \cdot u' - u = -4 \\ & \qquad u = 4 \\ & \qquad z^{-1} = 4 \\ & \qquad z = \frac{1}{4} \\ & \qquad y' = \frac{1}{4} \\ & \qquad y = \frac{1}{4}x \\ & \qquad y = -\frac{1}{2} \\ & \qquad \text{bod } \left(-2, -\frac{1}{2}\right) \\ u' + \frac{u}{x+2} &= -\frac{2x}{x+2} \end{aligned}$$

Vyřešíme homogenní rovnici

$$\begin{aligned}
u' + \frac{u}{x+2} &= 0 \quad / : u, \quad u \not\equiv 0 \\
& \quad\ u \equiv 0 \Rightarrow u' \equiv 0 \\
& \quad\ 0 + 0 = 0 \\
& \quad\ 0 = 0 \\
& \quad\ u \equiv 0 \text{ je řešení} \\
\frac{1}{u}u' + \frac{1}{x+2} &= 0 \\
\int \frac{1}{u}\,\mathrm{d}u + \int \frac{1}{x+2}\,\mathrm{d}x &= 0 \\
\ln|u| + \ln|x+2| &= c, \quad c \in \mathbb{R} \\
\ln|u| + \ln|x+2| &= c \cdot \ln \mathrm{e}, \quad c \in \mathbb{R} \\
\ln|u| + \ln|x+2| &= \ln \mathrm{e}^c, \quad c \in \mathbb{R} \\
|u| \cdot |x+2| &= \mathrm{e}^c, \quad c \in \mathbb{R} \\
|u| \cdot |x+2| &= c, \quad c > 0 \quad (\mathrm{e}^c = c) \\
|u| &= \frac{c}{|x+2|}, \quad c > 0 \\
u_H &= \frac{c}{x+2}, \quad c \neq 0 \\
u \equiv 0 \text{ bylo řešení} \Rightarrow u_H &= \frac{c}{x+2}, \quad c \in \mathbb{R}
\end{aligned}$$

Najdeme partikulární řešení metodou variace konstant

$$\begin{aligned}
u_P &= \frac{c(x)}{x+2} \\
u' &= \frac{c'(x+2) - c}{(x+2)^2} \\
\frac{c'(x+2) - c}{(x+2)^2} + \frac{\frac{c}{x+2}}{x+2} &= -\frac{2x}{x+2} \\
\frac{c'(x+2)}{(x+2)^2} - \frac{c}{(x+2)^2} + \frac{c}{(x+2)^2} &= -\frac{2x}{x+2} \\
\frac{c'}{x+2} &= -\frac{2x}{x+2} \quad / \cdot (x+2) \\
c' &= -2x \\
c &= -2\int x\,\mathrm{d}x = -x^2 \\
u_P &= -\frac{x^2}{x+2}
\end{aligned}$$

Získáváme řešení

$$\begin{aligned}
u &= u_H + u_P \\
u &= \frac{c}{x+2} - \frac{x^2}{x+2} = \frac{c - x^2}{x+2}, \quad c \in \mathbb{R}
\end{aligned}$$

Vrátíme substituci

$$\begin{aligned} z^{-1} &= u \\ \frac{1}{z} &= \frac{c - x^2}{x + 2}, \quad c \in \mathbb{R} \\ z &= \frac{x + 2}{c - x^2}, \quad c \in \mathbb{R} \end{aligned}$$

Vrátíme substituci

$$\begin{aligned} y' &= z \\ y' &= \frac{x + 2}{c - x^2}, \quad c \in \mathbb{R} \\ \int 1 \, \mathrm{d}y &= \int \frac{x + 2}{c_1 - x^2} \, \mathrm{d}x, \quad c_1 \in \mathbb{R} \\ y + c_2 &= \int \frac{x}{c_1 - x^2} \, \mathrm{d}x + 2 \int \frac{1}{c_1 - x^2} \, \mathrm{d}x, \quad c_1, c_2 \in \mathbb{R} \\ y + c_2 &= -\frac{1}{2} \int \frac{2x}{c_1 - x^2} \, \mathrm{d}x + 2 \int \frac{1}{c_1 - x^2} \, \mathrm{d}x, \quad c_1, c_2 \in \mathbb{R} \\ y + c_2 &= -\frac{1}{2} \ln |c_1 - x^2| + 2 \int \frac{1}{c_1 - x^2} \, \mathrm{d}x, \quad c_1, c_2 \in \mathbb{R} \end{aligned}$$

Pro $c_1 > 0$

$$\begin{aligned} \int \frac{1}{c_1 - x^2} \, \mathrm{d}x &= \frac{1}{c_1} \int \frac{1}{1 - \frac{x^2}{c_1}} \, \mathrm{d}x = \frac{1}{c_1} \int \frac{1}{1 - \left(\frac{x}{\sqrt{c_1}}\right)^2} \, \mathrm{d}x = \frac{1}{\sqrt{c_1}} \int \frac{1}{1 - t^2} \, \mathrm{d}t = \\ &= \frac{1}{\sqrt{c_1}} \int \frac{1}{(1 + t)(1 - t)} \, \mathrm{d}t = \frac{1}{\sqrt{c_1}} \left(\int \frac{1}{1 + t} \, \mathrm{d}t + \int \frac{1}{1 - t} \, \mathrm{d}t \right) = \\ &= \frac{1}{\sqrt{c_1}} (\ln |1 + t| - \ln |1 - t|) + k = \frac{1}{\sqrt{c_1}} \ln \left| \frac{1 + t}{1 - t} \right| + k = \\ &= \frac{1}{\sqrt{c_1}} \ln \left| \frac{1 + \frac{x}{\sqrt{c_1}}}{1 - \frac{x}{\sqrt{c_1}}} \right| + k = \frac{1}{\sqrt{c_1}} \ln \left| \frac{\sqrt{c_1} + x}{\sqrt{c_1} - x} \right| + k, \quad k = 0 \end{aligned}$$

Použili jsme substituci

$$\begin{aligned} \frac{x}{\sqrt{c_1}} &= t \\ \mathrm{d}x &= \sqrt{c_1} \, \mathrm{d}t \end{aligned}$$

Pro $c_1 < 0 \Rightarrow c_1 = -c_1', \quad c_1' > 0$

$$\begin{aligned} \int \frac{1}{c_1 - x^2} \, \mathrm{d}x &= \int \frac{1}{-c_1' - x^2} \, \mathrm{d}x = -\frac{1}{c_1'} \int \frac{1}{1 + \left(\frac{x}{\sqrt{c_1'}}\right)^2} \, \mathrm{d}x = \\ &= -\frac{1}{\sqrt{c_1'}} \operatorname{arctg} \frac{x}{\sqrt{c_1'}} + k, \quad k = 0 \end{aligned}$$

Použili jsme substituci

$$\begin{aligned}
\frac{x}{\sqrt{c_1'}} &= t \\
\mathrm{d}x &= \sqrt{c_1'}\,\mathrm{d}t
\end{aligned}$$

Pro $c_1 = 0$

$$\int \frac{1}{c_1 - x^2}\,\mathrm{d}x = \int \frac{1}{-x^2}\,\mathrm{d}x = \frac{1}{x} + k, \quad k = 0$$

Řešení diferenciální rovnice je

$$\begin{aligned}
y &= -\frac{1}{2}\ln|c_1 - x^2| + \frac{2}{\sqrt{c_1}}\ln\left|\frac{\sqrt{c_1} + x}{\sqrt{c_1} - x}\right| + c_2, \quad c_1 > 0, c_2 \in \mathbb{R}, x \neq -2 \\
y &= -\frac{1}{2}\ln|c_1 - x^2| - \frac{2}{\sqrt{-c_1}}\operatorname{arctg}\frac{x}{\sqrt{-c_1}} + c_2, \quad c_1 < 0, c_2 \in \mathbb{R}, x \neq -2 \\
y &= -\ln x + \frac{2}{x} + c_2, \quad c_1 = 0, c_2 \in \mathbb{R}, x \neq -2
\end{aligned}$$