Diferenciální rovnice 1. řádu

Lineární diferenciální rovnice 1. řádu

$$y' = \frac{x + 2y + 3}{2x + 5y + 6}$$

Řešení

$$\begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5 - 4 \neq 0$$

Determinant se nerovná nule, proto posuneme souřadnice

$$\begin{aligned} y' &= \frac{x + 2y + 3}{2x + 5y + 6} \\ x &= t + A \\ y &= u + B \\ \dot{u} &= y' \\ \dot{u} &= \frac{t + A + 2u + 2B + 3}{2t + 2A + 5u + 5B + 6} \\ \dot{u} &= \frac{t + 2u + A + 2B + 3}{2t + 5u + 2A + 5B + 6} \end{aligned}$$

Řešení rovnice

$$\begin{aligned} A + 2B + 3 = 0 &\qquad / \cdot (-2) \\ \underline{2A + 5B + 6 = 0} & \\ B = 0 & \\ \\ A + 3 = 0 & \\ A = -3 & \end{aligned}$$

Dostáváme rovnici

$$\begin{aligned} \dot{u} &= \frac{t + 2u}{2t + 5u} \\ \dot{u} &= \frac{t(1 + 2\frac{u}{t})}{t\left(2 + 5\frac{u}{t}\right)}, \quad t \neq 0 \end{aligned}$$

$$\begin{aligned} & t = 0 \Rightarrow \dot{u} = \frac{2}{5} \Rightarrow u = \frac{2}{5}t = 0 \Rightarrow y = 0 \\ & x = t - 3 = -3 \\ & \Rightarrow \text{bod}\,(-3, 0) \end{aligned}$$

$$
\begin{aligned}
\dot{u} &= \frac{1+2\frac{u}{t}}{2+5\frac{u}{t}}, \quad z=\frac{u}{t} \\
& \qquad\qquad u = zt \\
& \qquad\qquad \dot{u} = \dot{z}t + z \\
\dot{z}t + z &= \frac{1+2z}{2+5z} \\
\dot{z}t &= \frac{1+2z-2z-5z^2}{2+5z} \\
\dot{z}t &= \frac{1-5z^2}{2+5z} \quad / \cdot (2+5z) \quad / : t \neq 0 \quad / : \left(1-5z^2\right) \neq 0 \\
& \qquad\qquad 1-5z^2 = 0 \\
& \qquad\qquad 5z^2 = 1 \\
& \qquad\qquad z^2 = \frac{1}{5} \\
& \qquad\qquad z_{1,2} = \pm\frac{1}{\sqrt{5}} \\
\frac{\dot{z}\,(2+5z)}{1-5z^2} &= \frac{1}{t} \\
\int \frac{2+5z}{1-5z^2}\mathrm{d}z &= \int \frac{1}{t}\mathrm{d}t
\end{aligned}
$$

Nejprve vyřešíme integrál na levé straně

$$
\int \frac{2+5z}{1-5z^2}\mathrm{d}z = 2\int \frac{1}{1-5z^2}\mathrm{d}z + 5\int \frac{z}{1-5z^2}\mathrm{d}z
$$

$$
2\int \frac{1}{1-5z^2}\mathrm{d}z = 2\int \frac{1}{1-\left(\sqrt{5}z\right)^2}\mathrm{d}z = \frac{2}{\sqrt{5}}\int \frac{1}{1-s^2}\mathrm{d}s = \frac{1}{\sqrt{5}}\ln\left|\frac{1+\sqrt{5}z}{1-\sqrt{5}z}\right| + c, \quad c \in \mathbb{R}
$$

Substituce:

$$
\begin{aligned}
\sqrt{5}z &= s \\
\sqrt{5}\mathrm{d}z &= \mathrm{d}s \\
\mathrm{d}z &= \frac{1}{\sqrt{5}}\mathrm{d}s
\end{aligned}
$$

$$
5\int \frac{z}{1-5z^2}\mathrm{d}z = -\frac{5}{10}\int \frac{1}{s}\mathrm{d}s = -\frac{1}{2}\ln|s| + c = -\frac{1}{2}\ln\left|1-5z^2\right| + c, \quad c \in \mathbb{R}
$$

Substituce:

$$
\begin{aligned}
1-5z^2 &= s \\
-10z\mathrm{d}z &= \mathrm{d}s \\
z\mathrm{d}z &= -\frac{1}{10}\mathrm{d}s
\end{aligned}
$$

Nyní můžeme pokračovat ve výpočtu

$$
\begin{aligned}
\int \frac{2+5z}{1-5z^2}\mathrm{d}z &= \int \frac{1}{t}\mathrm{d}t \\
\frac{1}{\sqrt{5}}\ln\left|\frac{1+\sqrt{5}}{1-\sqrt{5}z}\right| - \frac{1}{2}\ln\left|1-5z^2\right| &= \ln|t| + c, \quad / \cdot 2 \quad c \in \mathbb{R} \\
\frac{2}{\sqrt{5}}\ln\left|\frac{1+\sqrt{5}z}{1-\sqrt{5}z}\right| - \ln\left|1-5z^2\right| &= \ln|t|^2 + c, \quad c \in \mathbb{R} \qquad (c = 2c) \\
\ln\frac{\left(\frac{1+\sqrt{5}z}{1-\sqrt{5}z}\right)^{\frac{2}{\sqrt{5}}}}{|1-5z^2|} &= \ln t^2 + \ln c, \quad c > 0 \quad (c = \mathrm{e}^c) \\
\ln\frac{\left(1+\sqrt{5}z\right)^{\frac{2}{\sqrt{5}}}}{\left(1-\sqrt{5}z\right)^{\frac{2}{\sqrt{5}}}|1-5z^2|} &= \ln c\,t^2, \quad c > 0 \\
\frac{\left(1+\sqrt{5}z\right)^{\frac{2}{\sqrt{5}}}}{\left(1-\sqrt{5}z\right)^{\frac{2}{\sqrt{5}}}|1-5z^2|} &= c\,t^2, \quad c > 0 \\
\frac{\left(1+\sqrt{5}z\right)^{\frac{2}{\sqrt{5}}}}{\left(1-\sqrt{5}z\right)^{\frac{2}{\sqrt{5}}}(1-5z^2)} &= c\,t^2, \quad c \neq 0
\end{aligned}
$$

Vrátíme zpět substituce

$$
\begin{aligned}
\frac{\left(1+\sqrt{5}\frac{u}{t}\right)^{\frac{2}{\sqrt{5}}}}{\left(1-\sqrt{5}\frac{u}{t}\right)^{\frac{2}{\sqrt{5}}}\left(1-5\left(\frac{u}{t}\right)^2\right)} &= c\,t^2, \quad c \neq 0, \quad z = \frac{u}{t} \\
\frac{\left(1+\frac{\sqrt{5}y}{x+3}\right)^{\frac{2}{\sqrt{5}}}}{\left(1-\frac{\sqrt{5}y}{x+3}\right)^{\frac{2}{\sqrt{5}}}\left(1-5\left(\frac{y}{x+3}\right)^2\right)} &= c\,(x+3)^2, \quad c \neq 0, \quad x = t-3 \Rightarrow t = x+3, \quad y = u \\
\frac{\left(1+\frac{\sqrt{5}y}{x+3}\right)^{\frac{2}{\sqrt{5}}}}{(x+3)^2\left(1-\frac{\sqrt{5}y}{x+3}\right)^{\frac{2}{\sqrt{5}}}\left(1-5\left(\frac{y}{x+3}\right)^2\right)} &= c, \quad c \neq 0 \\
\frac{\frac{\left(x+3+\sqrt{5}y\right)^{\frac{2}{\sqrt{5}}}}{(x+3)^{\frac{2}{\sqrt{5}}}}}{\frac{(x+3)^2\left(x+3-\sqrt{5}y\right)^{\frac{2}{\sqrt{5}}}\left((x+3)^2-5y^2\right)}{(x+3)^{\frac{2}{\sqrt{5}}}(x+3)^2}} &= c, \quad c \neq 0 \\
\frac{\left(x+3+\sqrt{5}y\right)^{\frac{2}{\sqrt{5}}}}{\left(x+3-\sqrt{5}y\right)^{\frac{2}{\sqrt{5}}}\left((x+3)^2-5y^2\right)} &= c, \quad c \neq 0
\end{aligned}
$$

Singulárními řešeními jsou

$$
\begin{aligned}
z &= \pm\frac{1}{\sqrt{5}} \\
\frac{u}{t} &= \pm\frac{1}{\sqrt{5}} \\
\frac{y}{x+3} &= \pm\frac{1}{\sqrt{5}} \\
y &= \pm\frac{1}{\sqrt{5}}\,(x+3)\,, \quad x \in \mathbb{R}
\end{aligned}
$$