Diferenciální rovnice 1. řádu

Zlomkový typ

$$y' = 2\left(\frac{y+4}{x+y-5}\right)^2$$

Řešení

$$\begin{vmatrix} 0 & 1 \\ 1 & 1 \end{vmatrix} = -1 \neq 0 \Rightarrow \text{ Posun souřadnic}$$

$$\begin{aligned} t + A &= x \\ u + B &= y \\ \dot{u} &= y' \end{aligned}$$

$$\begin{aligned} \dot{u} &= 2\left(\frac{u+B+4}{t+A+u+B-5}\right)^2 \\ \dot{u} &= 2\left(\frac{u+B+4}{t+u+A+B-5}\right)^2 \end{aligned}$$

Soustava rovnic:

$$\begin{aligned} B + 4 = 0 \\ \underline{A + B - 5 = 0} \\ \underline{\underline{B = -4}} \\ A - 4 - 5 = 0 \\ \underline{\underline{A = 9}} \end{aligned}$$

$$\begin{aligned} x &= t + 9 \\ y &= u - 4 \end{aligned}$$

$$\dot{u} = 2\left(\frac{u}{t+u}\right)^2$$

$?t = 0$

$t = 0 \Rightarrow \dot{u} = 2; y' = \dot{u} \Rightarrow y' = 2$

$y = 2x$

$t = 0 \Rightarrow x = 9 \Rightarrow y = 18$

bod $(9, 18)$

$$\dot{u} = 2\left(\frac{t\frac{u}{t}}{t\left(1+\frac{u}{t}\right)}\right)^2 \qquad t \neq 0$$

$$\dot{u} = 2\left(\frac{\frac{u}{t}}{1+\frac{u}{t}}\right)^2$$

Použijeme substituci:

$$
\begin{aligned}
\frac{u}{t} &= z \\
u &= zt \\
\dot{u} &= \dot{z}t + z
\end{aligned}
$$

Získáme:

$$
\begin{aligned}
\dot{z}t + z &= 2\left(\frac{z}{1+z}\right)^2 && \\
\dot{z}t + z &= \frac{2z^2}{1+2z+z^2} && / - z \\
\dot{z}t &= \frac{2z^2 - z - 2z^2 - z^3}{1+2z+z^2} && \\
\dot{z}t &= \frac{-z-z^3}{1+2z+z^2} && / \cdot (1+2z+z^2) \\
& && / : t \neq 0 \\
& && / : (-z-z^3) \neq 0 \\
& && \\
& && (-z-z^3) = -z(1+z^2) \\
& && (1+z^2) \neq 0 \\
& && ?z = 0 \Rightarrow \dot{z} = 0 \\
& && 0 \cdot t = \frac{0}{1+2\cdot 0+0^2} \\
& && 0 = 0 \\
& && z \equiv 0 \\
& && z = \frac{u}{t} \\
& && \text{u} \equiv 0 \text{ je řešení} \\
\frac{(1+2z+z^2)\dot{z}}{-z-z^3} &= \frac{1}{t} &&
\end{aligned}
$$

Integrujeme

$$
\int \frac{(1+2z+z^2)}{-z-z^3}\,\mathrm{d}z = \int \frac{1}{t}\,\mathrm{d}t
$$

1. integrál

$$
\int \frac{(1+2z+z^2)}{-z-z^3}\,\mathrm{d}z
$$

Parciální zlomky

$$\begin{aligned}\frac{(z^2+2z+1)}{z(-z^2-1)} &= \frac{Az+B}{-z^2-1}+\frac{C}{z} & / \cdot z(-z^2-1)\\ (z^2+2z+1) &= z(Az+B)+C(-z^2-1)\\ (z^2+2z+1) &= Az^2+Bz-Cz^2-C\\ (z^2+2z+1) &= (A+C)z^2+Bz-C\end{aligned}$$

$$\begin{aligned}A &= 0\\ B &= 2\\ C &= 1\end{aligned}$$

$$-\int \frac{2}{-z^2-1}\,\mathrm{d}z - \int \frac{1}{z}\,\mathrm{d}z = 2\mathrm{arctg}\, z - \ln|z| + c \quad ; c \in \mathbb{R}$$

2. integrál

$$\int \frac{1}{t}\,\mathrm{d}t = \ln|t| + c \quad ; c \in \mathbb{R}$$

Získáme:

$$\begin{aligned}2\mathrm{arctg}\, z - \ln|z| &= \ln|t| + c & ; c \in \mathbb{R}\\ \ln \mathrm{e}^{2\mathrm{arctg}\, z} - \ln|z| &= \ln|t| + c & ; c \in \mathbb{R}\\ & & / - \ln|t|\\ \ln \mathrm{e}^{2\mathrm{arctg}\, z} - \ln|z| - \ln|t| &= c & ; c \in \mathbb{R}\\ & & / \cdot (-1)\\ -\ln \mathrm{e}^{2\mathrm{arctg}\, z} + \ln|z| + \ln|t| &= c & ; c \in \mathbb{R}\\ & & (c = -c)\\ \ln|zt| + \ln\frac{1}{\mathrm{e}^{2\mathrm{arctg}\, z}} &= \ln c & ; c > 0\\ \ln\frac{|zt|}{\mathrm{e}^{2\mathrm{arctg}\, z}} &= \ln c & ; c > 0\\ \frac{|zt|}{\mathrm{e}^{2\mathrm{arctg}\, z}} &= c & ; c > 0\\ \frac{zt}{\mathrm{e}^{2\mathrm{arctg}\, z}} &= c & ; c \neq 0\end{aligned}$$

Vracím 1. substituci:

$$\begin{aligned}\frac{\frac{u}{t}t}{\mathrm{e}^{2\mathrm{arctg}\,\frac{u}{t}}} &= c & ; c \neq 0\\ \frac{u}{\mathrm{e}^{2\mathrm{arctg}\,\frac{u}{t}}} &= c & ; c \neq 0\end{aligned}$$

$u \equiv 0$ bylo řešení – získáme je volbou $c = 0$

$$\frac{u}{\mathrm{e}^{2\,\mathrm{arctg}\,\frac{u}{t}}} = c \qquad ; c \in \mathbb{R}$$

Vracím 2. substituci:

$$\frac{y+4}{\mathrm{e}^{2\,\mathrm{arctg}\,\frac{y+4}{x-9}}} = c \qquad ; c \in \mathbb{R}$$

$$c = \frac{y+4}{\mathrm{e}^{2\,\mathrm{arctg}\,\frac{y+4}{x-9}}} \qquad ; c \in \mathbb{R}$$